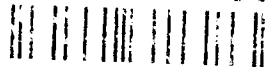
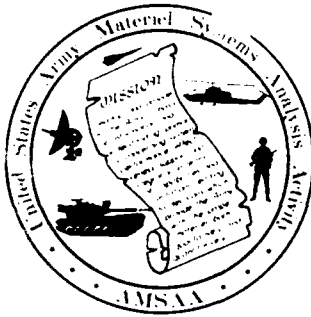


AD-A243 519



2



AMSAA

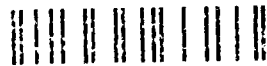
TECHNICAL REPORT NO. 501

THE INSTABILITY OF
LINEAR HETEROGENEOUS LANCHESTER EQUATIONS

HERBERT E. COHEN

APPROVED FOR PUBLIC RELEASE; DISTRIBUTION IS UNLIMITED.

91-18312



U. S. ARMY MATERIEL SYSTEMS ANALYSIS ACTIVITY
ABERDEEN PROVING GROUND, MARYLAND

DISPOSITION

When this report is no longer needed, Department of the Army organizations will destroy it in accordance with the procedures given in AR 380-5. Navy and Air Force elements will destroy it in accordance with applicable directives. Department of Defense contractors will destroy the report according to the requirements of Section 14 of the Industrial Security Manual for Safe-guarding Classified Information. All others will return the report to the U.S. Army Materiel Systems Analysis Activity, Aberdeen Proving Ground, Maryland 21005-5071.

DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position unless so specified by other official documentation.

WARNING

Information and data contained in this document are based on the input available at the time of preparation. The results may be subject to change and should not be construed as representing the AMC position unless so specified.

TRADE NAMES

The use of trade names in this report does not constitute an official endorsement or approval of the use of such commercial hardware or software. The report may not be cited for purposes of advertisement.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE				Form Approved OMB No. 0704-0188	
1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY			3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for Public Release; Distribution Unlimited		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE					
4. PERFORMING ORGANIZATION REPORT NUMBER(S)			5. MONITORING ORGANIZATION REPORT NUMBER(S)		
6a. NAME OF PERFORMING ORGANIZATION U.S. Army Materiel Systems Analysis Activity		6b. OFFICE SYMBOL (If applicable) AMXSY-MP	7a. NAME OF MONITORING ORGANIZATION		
6c. ADDRESS (City, State, and ZIP Code) Aberdeen Proving Ground, MD 21005-5071			7b. ADDRESS (City, State, and ZIP Code)		
8a. NAME OF FUNDING / SPONSORING ORGANIZATION		8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER		
8c. ADDRESS (City, State, and ZIP Code)			10. SOURCE OF FUNDING NUMBERS		
			PROGRAM ELEMENT NO	PROJECT NO.	TASK NO.
					WORK UNIT ACCESSION NO
11. TITLE (Include Security Classification) The Instability of Linear Heterogeneous Lanchester Equations					
12. PERSONAL AUTHOR(S) Herbert E. Cohen					
13a. TYPE OF REPORT Technical Report		13b. TIME COVERED FROM _____ TO _____		14. DATE OF REPORT (Year, Month, Day) 1991 November 20	
15. PAGE COUNT					
16. SUPPLEMENTARY NOTATION					
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB-GROUP	Combat Models, Lanchester Equations, Stability Eigen Values		
19. ABSTRACT (Continue on reverse if necessary and identify by block number)					
It is shown that the structure of the linear Lanchester attrition matrix leads to invariant properties associated with the sum of the eigenvalues and to the conclusion the linear Lanchester equations are inherently unstable.					
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT <input type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS			21. ABSTRACT SECURITY CLASSIFICATION		
22a. NAME OF RESPONSIBLE INDIVIDUAL			22b. TELEPHONE (Include Area Code)		22c. OFFICE SYMBOL

ACKNOWLEDGEMENT

The U.S. Army Materiel Systems Analysis Activity (AMSAA) recognizes the following individual for contributing to this report:

Peer Reviewer: Rich Sandmeyer, Combat Support Division (CSD).

Accession Year	
Dist	General
Dist	Tab
Justification	
By	
Distribution/	
Availability Codes	
Avail and/or	
Dist	Special
A-1	

The next page is blank.

THE INSTABILITY OF LINEAR HETEROGENEOUS LANCHESTER EQUATIONS

Invariant properties of differential equations provide useful information on the nature of the solutions. The linear Lanchester equations have interesting invariant properties which this writer believes are not readily known within the community.

An n-dimensional linear time varying Lanchester equation [1] can be described by

$$\dot{x} = F(t) x(t), \quad x(t_0) \quad (1)$$

where $F(t)$ is a nxn matrix in R and can be represented by a partitioned matrix of the form

$$F = \begin{pmatrix} \underline{0} & -A \\ -B & \underline{0} \end{pmatrix}$$

where the elements of F , $[f_{ij}]$, are all less than 1 and x is a n-dimensional "force strength" vector in R^n and t_0 is the initial time. Introducing the fundamental matrix $\phi(t, t_0)$ [2-8] with the property

$$x(t) = \phi(t, t_0) x(t_0)$$

it can be shown that the

$$\det \phi(t, t_0) = e^{\int_{t_0}^t \text{tr } F(t) dt} \quad (2)$$

Since the trace of F , $\text{tr } (F(t))$, is zero for all t , it follows that

$$\det \phi(t, t_0) = 1 \quad (3)$$

for all n-dimension linear Lanchester equations.

Since in many cases in practice, $F(t)$, is taken as constant or piecewise constant over a sufficiently small intervals $t_i \leq t \leq t_{i+1}$, it follows from systems theory that

$$\Phi(t, t_0) = T^{-1} e^{Dt} T \quad (4)$$

where T is a similarity transformation and D is a diagonal matrix of eigenvalues for $F(t)$ in $t_i \leq t \leq t_{i+1}$, assuming distinct eigenvalues. Thus, it follows that the

$$\begin{aligned} \det \Phi(t, t_0) &= \det(e^{Dt}) \\ &= e^{(\sum \lambda_i)t} \end{aligned}$$

Since $\det \Phi(t, t_0) = 1$ we can conclude that

$$\sum_{i=1}^n \lambda_i = 0 \quad (5)$$

for all linear Lanchester equations. It is this property on the sum of the eigenvalues which is invariant for linear Lanchester equations. It should be noted that Eq (5) is also true when the eigenvalues are not distinct.

The analysis of Equation (5) leads one to conclude that either all eigenvalues reside on the imaginary axis resulting in harmonic oscillation or there exists at least one eigenvalue residing in the right half plane so that our system is inherently unstable.

When one applies a finite difference approximation to the derivative in Eq (1) we obtain

$$X(t_i + 1) = (I + F(t_i)) X(t_i) \quad (6)$$

It then follows that the sum of the eigenvalues (ζ_i), associated with the difference equation, Eq (6), is

$$\sum_{i=1}^n \zeta_i = n \quad (7)$$

where n is the dimension of x .

For Eq (6) to be stable all eigenvalues must lie in the unit circle so that $|\zeta_i| < 1$, otherwise it is unstable. The eigenvalues ζ_i associated with Eq (6) are related to the eigenvalues λ_i of the continuous case by

$$\zeta_i = 1 + \lambda_i$$

so that if the continuous problem is nonstable or have all of its eigenvalues on the imaginary axis, then the difference equation is inherently unstable.

It is appropriate at this point to present two examples which demonstrate this invariant property:

Case 1: Linear Lanchester equation of order two. This is given by the simple equation

$$\dot{x}_1 = -\alpha x_2$$

$$\dot{x}_2 = -\beta x_1$$

$$\text{or} \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

which for $x = (x_1, x_2)^T$ we have

$$\dot{x} = Fx$$

$$\text{with} \quad F = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix}$$

and α, β real and less than one.

The matrix F has eigenvalues

$$\lambda_+ = + \sqrt{\alpha \beta}$$

and

$$\lambda_- = -\sqrt{\alpha \beta}$$

so that the 2nd order system is inherently unstable. Similarly the corresponding difference equation will also be unstable since it has an eigenvalue of value

$$1 + \sqrt{\alpha \beta} \text{ which is outside the unit circle.}$$

Case 2: Linear Lanchester equation of n th order ($n=50$). This equation is of the form

$$\dot{X} = FX$$

where F is 50×50 and represented

$$\text{by } F = \begin{pmatrix} Q & -A \\ -B & Q \end{pmatrix}$$

with A and B each being a 25×25 matrix whose elements are randomly generated by Matlab and less than one. This means that x is a vector of dimension 50. Using Matlab for the continuous case, it is found that

$$\sum_{i=1}^{50} \lambda_i = 1.3 \times 10^{-14}$$

with eigenvalues in the complex plane as shown in figure (1). It is immediately seen that λ_i have several eigenvalues in the right half plane so that this system is inherently unstable.

The finite difference equation for this problem yielded

$$\sum_{i=1}^{50} \zeta_i = 50$$

consistent with our analysis and has its eigenvalues in the complex plane as shown in figure (2). Please note the scales in both figures are different. However, there are several eigenvalues lying outside the unit circle so that this difference equation is inherently unstable.

In conclusion, we have seen that the structure of the linear heterogeneous Lanchester equations leads to an invariant condition associated with the sum of the eigenvalues with solutions which are inherently unstable or oscillate sinusoidally.

Figure 1. Eigenvalues in Complex Plane.
(Continuous Case)

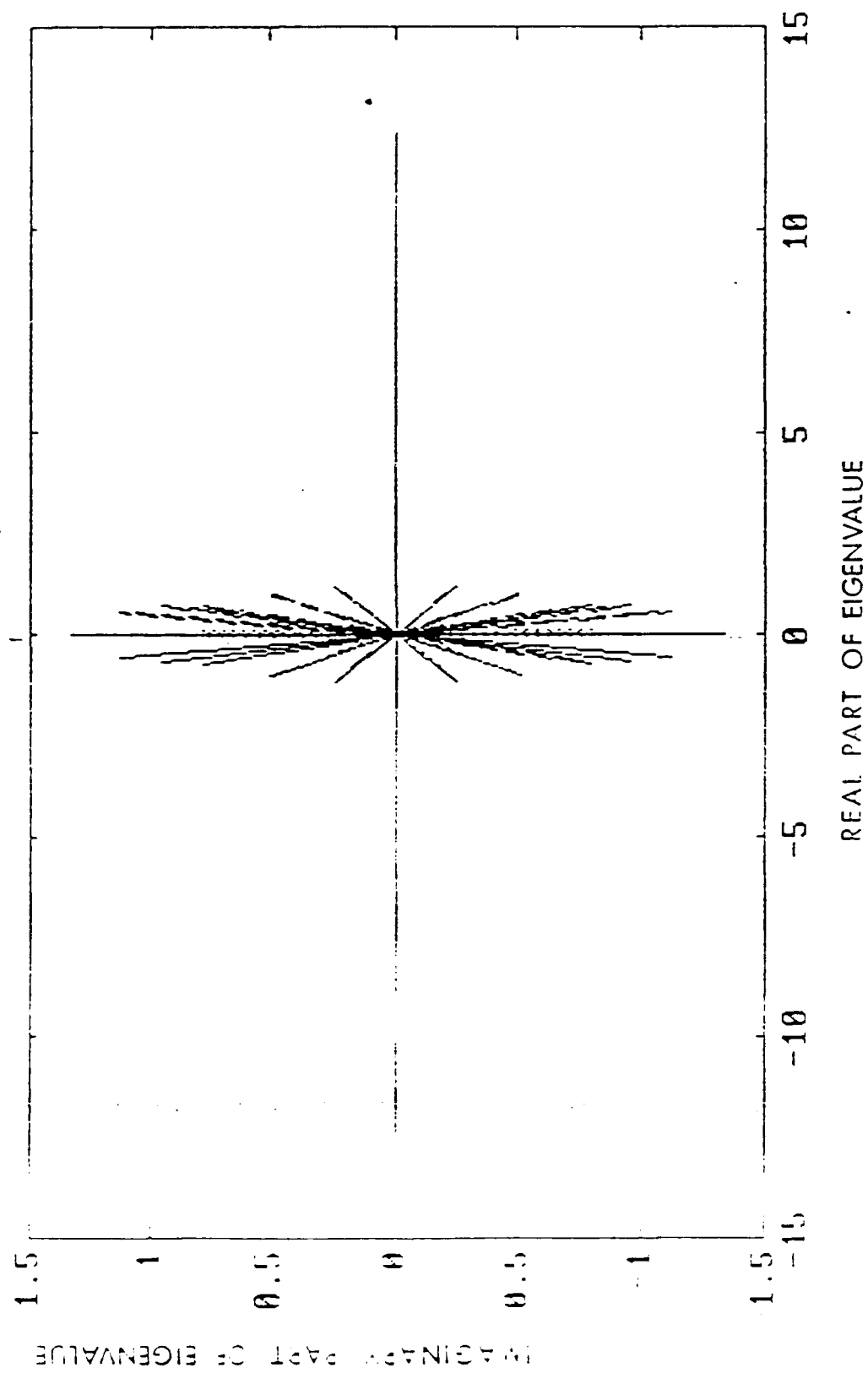
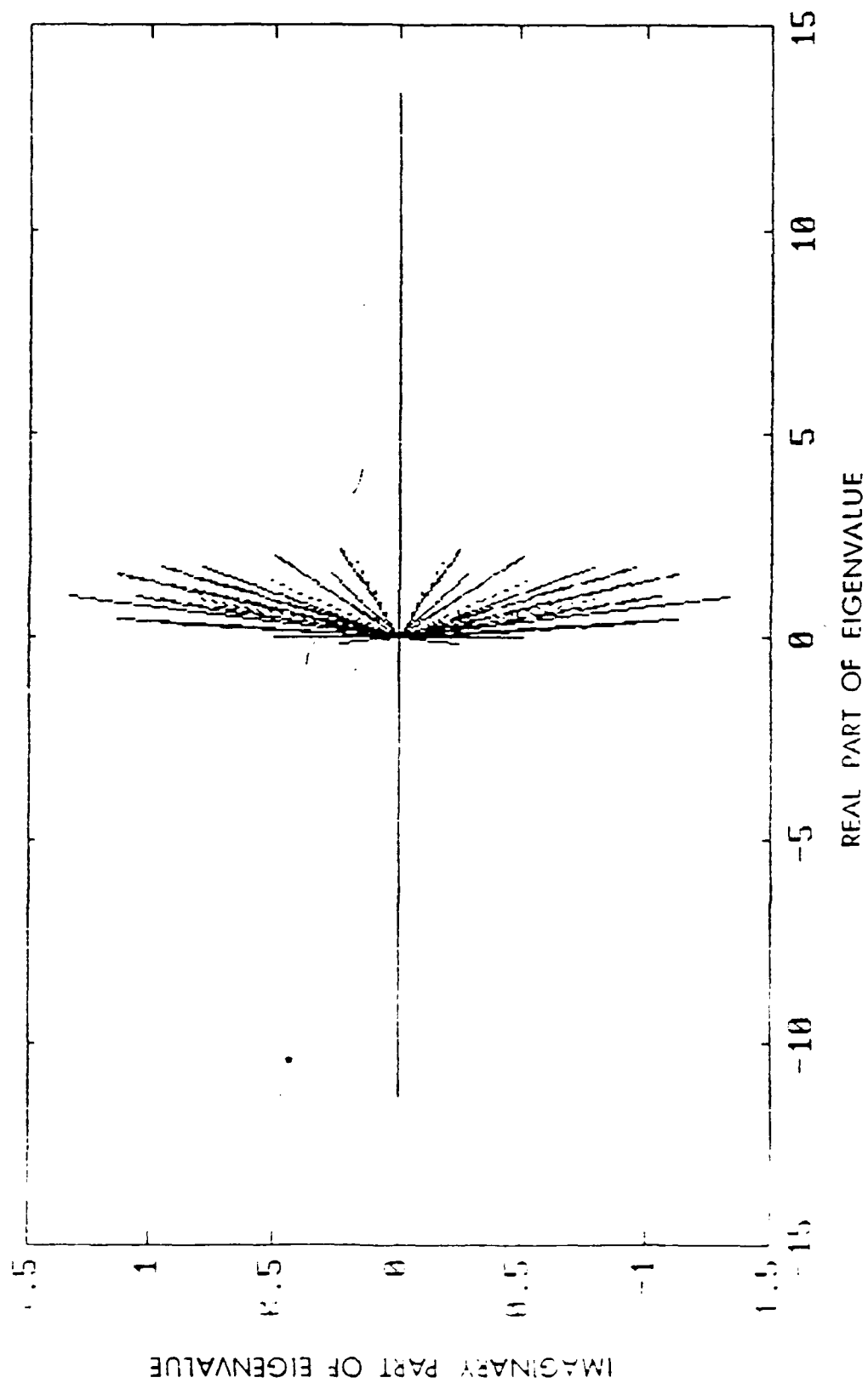


Figure 2. Eigenvalues in Complex Plane.
(Discrete Case)



REFERENCES

- [1] DARCOM-P, 706-102.
- [2] Coddington, E. A. and Levinson, N., Theory of Ordinary Differential Equations (1955), McGraw Hill.
- [3] Bellman, Richard, Introduction to Matrix Analysis (1960), McGraw Hill.
- [4] Gantmacher, F. R., The Theory of Matrices, Volume 1 and 2 (1959), Chelsea Publishing Company.
- [5] Zadeh, L. A., and Desoer, Charles A., Linear Systems Theory (1963), McGraw Hill.
- [6] Chen, C. T., Introduction to Linear System Theory (1970), Holt, Rinehart, and Winston, Inc.
- [7] Ogata, K., State Space Analysis of Control Systems (1967), Prentice-Hall.
- [8] Athans, M. and Falb, P. L., Optimal Control (1966), McGraw Hill.

DISTRIBUTION LIST

No. of Copies	Organization	No. of Copies	Organization
7	Commander Defense Technical Information Center ATTN: DTIC-FDAC Cameron Station, Bldg 5 Alexandria, VA 22304-6145	2	Commander U.S. Army Development and Employment Agency ATTN: MODE-TED-SAB (2 cys) Fort Lewis, WA 98433-5000
2	Commander U.S. Army Materiel Command ATTN: AMXPA AMCDMA-M 5001 Eisenhower Avenue Alexandria, VA 22333-0001	4	Director U.S. Army TRADOC Systems ATTN: ATOR-TSL ATOR-T ATOR-TE ATOR-TF White Sands Missile Range, NM 88002-5502
1	Director Combat Data Information Center AFWAL/FIES/CDIC Wright-Patterson AFB, OH 45433-5000	1	Commandant U.S. Army Infantry School ATTN: ATSH-CD-CS-OR Fort Benning, GA 31905-5400
1	Pentagon Library ATTN: AN-AL-RS (Army Studies) Pentagon, Room 1A518 WASH, DC 20310	3	Commander U.S. Army Tank-Automotive Command ATTN: AMSTA-Z (Tech Lib) AMSTA-ZE AMSTA-ZSA Warren, MI 48090
1	Commander U.S. Army Concepts Analysis Agency ATTN: CSCA-MSL-L 8120 Woodmont Avenue Bethesda, MD 20814-2797	1	HQDA (DACS-CV) WASH, DC 20310
1	Dr. Clint V. Ancker, Jr. 23908 Malibu Knolls Road Malibu, CA 90265	2	Project Manager M1E1 Tank System ATTN: AMCPM-M1E1 Warren, MI 48090
1	Director U.S. Army TRADOC Analysis Command ATTN: ATRC-FMD (Mr. E. Arendt) Fort Leavenworth, KS 66027-5200	1	Director of Research U.S. Army TRADOC Analysis Command ATTN: ATRC-RD (Ms. R. Baird) White Sands Missile Range, NM 88002-5502

DISTRIBUTION LIST (Continued)

No. of Copies	Organization	No. of Copies	Organization
1	Systems Assessment Group Royal Military College of Science, Mr. M. Bathe Shrivenham Swindon Wilts SN6 8LA United Kingdom	1	Colonel C. E. Evans Director Army Model Improvement Program Management Office ATTN: ATZL-CAN-DO Fort Leavenworth, KS 66027-5000
1	Commander Joint Warfare Center ATTN: Mr. E. Blair, Technical Advisor Hurlburst Field Eglin Air Force Base, FL 32544-5000	1	Robert Farrell Vector Research Incorporated PO Box 1506 Ann Arbor, MI 48106
1	Director U.S. Army TRADOC Analysis Command ATTN: ATRC-TD (Mr. F. Campbell) Fort Leavenworth, KS 66027-5200	1	HQDA (SARD-TNT) ATTN: LTCOL R. Fejfar Pentagon WASH, DC 20310-0105
1	Director U.S. Army TRADOC Analysis Command Requirements and Programs Directorate ATTN: ATRC-RP (Mr. K. Carson) Fort Monroe, VA 23651-5143	1	Mr. David Finley General Dynamics, Fort Worth Division PO Box 748 Mail Zone 4091 Fort Worth, TX 76101
1	Director U.S. Army TRADOC Analysis Command ATTN: ATRC-W (Dr. D. Collier) White Sands Missile Range, NM 88002-5502	1	Director Army Model Improvement Program, Management Office ATTN: ATZL-CAN-DO (MAJ Halek) Fort Leavenworth, KS 66027-5000
1	Professor Ralph Disney Department of Industrial Engineering Texas A&M University College Station, TX 77843	1	Defense Operational Analysis Establishment ATTN: Mr. Geoff Hawkins Broadoaks, Parvis Road West Byfleet Weybridge Surrey KT14 6LY United Kingdom
1	Colonel T.N. Dupuy, USA Ret. Data Memory Systems, Inc. 10392 Democracy Lane Fairfax, VA 22030	1	U.S. Army Concepts Analysis Agency ATTN: CSCA-MVM (Dr. R. Hembold) 8210 Woodmont Avenue Bethesda, MD 20814-2797

DISTRIBUTION LIST (Continued)

No. of Copies	Organization	No. of Copies	Organization
1	U.S. Army Concepts Analysis Agency ATTN: CSCA-RSR (Prof. L. Ingber) 8120 Woodmont Avenue Bethesda, MD 20814-2797	1	Director U.S. Army TRADOC Analysis Command ATTN: ATRC-DPS (Dr. H. Sassenfeld) White Sands Missile Range, NM 88002-5502
1	U.S. Army Tank-Automotive Command ATTN: AMSTA-RSA (Dr. W. Jackson) Warren, MI 48397-5000	1	Director U.S. Army TRADOC Analysis Command ATTN: ATRC-WC (P. Shugart) White Sands Missile Range, NM 88002-5502
1	Commander HQDA ATTN: DAMO-ZDS (Dr. J. Metzger) WASH, DC 20310	1	U.S. Army Concepts Analysis Agency ATTN: CSCA-RQN (MAJ Staniec) 8120 Woodmont Avenue Bethesda, MD 20814
1	Dr. Louis Moore RAND Corporation 1700 Main Street Santa Monica, CA 90406-2138	1	Professor James Taylor Department of Operations Research Naval Postgraduate School Department 55 (Tw) Monterey, CA 93943-5000
1	Director U.S. Army TRADOC Analysis Command ATTN: ATRC-WEA (Dr. M. Parish) White Sands Missile Range, NM 88002-5502	1	Mr. David Thompson Vector Research Incorporated PO Box 1506 Ann Arbor, MI 48106
1	Professor Sam Parry Commander U.S. Naval Postgraduate School Code 55PY Monterey, CA 93943	1	Dr. Ralph M. Toms Development Projects Manager Conflict Simulation Laboratory, L-315 Lawrence Livermore National Laboratory PO Box 808 Livermore, CA 94550
1	Director U.S. Army TRADOC Analysis Command ATTN: ATRC-WEA (R. Porter) White Sands Missile Range, NM 88002-5502		

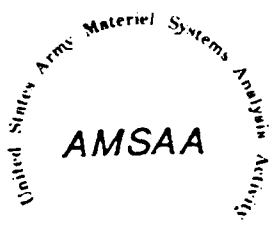

DISTRIBUTION LIST (Continued)

No. of Copies	Organization	No. of Copies	Organization
1	U.S. Army Concepts Analysis Agency ATTN: CSCA-RSA (J. Warren) 8120 Woodmont Avenue Bethesda, MD 20814	1	Mr. Patrick Wood Ministry of Defense (PE) CAI RARDE Fort Halsheard Sevenoaks, Kent TN 14 FBP United Kingdom
1	U.S. Army TRADOC Analysis Command ATTN: ATRC-RD (MAJ Welo) Director of Research White Sands Missile Range, NM 88002-5502	1	U.S. Army Concepts Analysis Agency ATTN: CSCA-RQN (MAJ M. Youngren) 8120 Woodmont Avenue Bethesda, MD 20814
1	HQDA ATTN: SAUS-OR (Dr. D. Willard) WASH, DC 20310-0102	1	General Dynamics Land Systems Division ATTN: Mr. J. O'Rourke PO Box 1800 Warren, MI 48090

Aberdeen Proving Ground

1	Director U.S. Army Human Engineering Laboratory ATTN: SLCH-SP (Dr. B. Cummings) Aberdeen Proving Ground, MD 21005-5001	1	Director U.S. Army Ballistics Research Laboratory ATTN: AMXBR-OD-ST, Bldg 305 Aberdeen Proving Ground, MD 21005-5066
2	Director U.S. Army Ballistics Research Laboratory ATTN: AMXBR-OD, Bldg 328 AMXBR-SECAD Aberdeen Proving Ground, MD 21005-5066	10	Director U.S. Army Materiel Systems Analysis Activity ATTN: AMXSY-CC (R. Sandmeyer) AMXSY-G AMXSY-GA AMXSY-GB AMXSY-C AMXSY-DA (H. Cohen) AMXSY-PA (RPC, 3 cys) Aberdeen Proving Ground, MD 21005-5071

GIST

 <p>AMSAA</p>	<p>TYPE OF REPORT: Technical Report</p> <p>REPORT NUMBER: 501</p> <p>REPORT DATE: November 1991</p>	
--	---	---

TITLE: The Instability of Linear Heterogeneous Lanchester Equations

REASON FOR PERFORMING THIS EFFORT: Investigation of the properties of the linear heterogeneous Lanchester equations which are used extensively by Army combat modelers.

MAIN OBJECTIVES OF THE EFFORT: Investigate invariant properties of linear heterogeneous Lanchester equations of combat and determine the potential impact on combat models.

SCOPE OF THE EFFORT: Apply systems theory to linear heterogeneous Lanchester equations in state variables and introduce properties of the state transition matrix to develop invariant properties of these equations.

CONTRACTED ADVISORY AND ASSISTANCE SERVICES: None.

SPONSOR: None.

PERFORMING ORGANIZATION/POINT OF CONTACT: U.S. Army Materiel Systems Analysis Activity (AMSAA), Special Studies and Activities Office, Herbert E. Cohen, DSN 298-2785.

DTIC/DLSIE ACCESSION NUMBER: Report available by contacting AMSAA's Reports Processing Center, DSN 298-4661.